

2-Dimensional Elastic Collisions without Trigonometry

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This document is intended to introduce you to solving 2-dimensional elastic collision problems for circles without complicated trigonometry. It is much easier to use vectors to solve 2-dimensional collision problems than to use trigonometry.

If you are unfamiliar with vectors, a web search for "vectors" should yield a variety of introductions. Be aware that vectors are commonly represented in two ways. One way is to specify a magnitude and an angle. The other is to specify the components (typically in the x and y directions). The two forms represent the same information and can be interconverted, but for some tasks one form may be more convenient than the other. Trigonometry is important for understanding the concept of vectors. However, the goal of this document is to avoid tedious (and computationally expensive) trigonometry. Therefore we will only use the component representation of a vector, not the magnitude and angle form. This reduces the actual computations to addition, subtraction, multiplication, division, and square roots.

Notation: Throughout this document, m is mass and v is velocity. Subscripts 1 and 2 distinguish between the two colliding objects. An apostrophe after a variable means that the value is taken *after* the collision (called *prime*; i.e., v' is "v prime").

Physics of elastic collisions in one dimension

An elastic collision is a collision in which kinetic energy is conserved. That means no energy is lost as heat or sound during the collision. In the real world, there are no perfectly elastic collisions on an everyday scale of size. But you can get the sense of an elastic collision by imagining a perfect pool ball which doesn't waste any energy when it collides. In an elastic collision, both kinetic energy and momentum are conserved (the total before and after the collision remains the same).

Momentum is the product of mass and velocity: $p = mv$

The kinetic energy of an object is one-half times its mass times the square of its velocity:

$$KE = \frac{1}{2}mv^2$$

Now it is easy to write the conservation of momentum and kinetic energy as two equations:

Conservation of momentum: $m_1 v_1 + m_2 v_2 = m_1 v'_1 + m_2 v'_2$

Conservation of kinetic energy: $\frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 = \frac{1}{2} m_1 v'^2_1 + \frac{1}{2} m_2 v'^2_2$

Combining these two equations and doing a lot of algebra gives the final (after collision) velocities of objects 1 and 2:

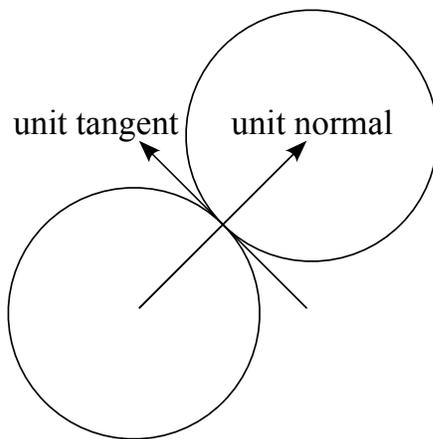
$$v'_1 = \frac{v_1(m_1 - m_2) + 2m_2 v_2}{m_1 + m_2} \quad v'_2 = \frac{v_2(m_2 - m_1) + 2m_1 v_1}{m_1 + m_2}$$

This result allows us to find the velocity of two objects after undergoing a one-dimensional elastic collision. We will use this result later in the 2-dimensional case.

Elastic collisions in two dimensions

We will follow a 7-step process to find the new velocities of two objects after a collision. The basic goal of the process is to project the velocity vectors of the two objects onto the vectors which are normal (perpendicular) and tangent to the surface of the collision. This gives us a normal component and a tangential component for each velocity. The tangential components of the velocities are not changed by the collision because there is no force along the line tangent to the collision surface. The normal components of the velocities undergo a one-dimensional collision, which can be computed using the one-dimensional collision formulas presented above. Next the unit normal vector is multiplied by the scalar (plain number, not a vector) normal velocity after the collision to get a vector which has a direction normal to the collision surface and a magnitude which is the normal component of the velocity after the collision. The same is done with the unit tangent vector and the tangential velocity component. Finally the new velocity vectors are found by adding the normal velocity and tangential velocity vectors for each object.

1. Find unit normal and unit tangent vectors. The unit normal vector is a vector which has a magnitude of 1 and a direction that is normal (perpendicular) to the surfaces of the objects at the point of collision. The unit tangent vector is a vector with a magnitude of 1 which is tangent to the circles' surfaces at the point of collision.



First find a normal vector. This is done by taking a vector whose components are the difference between the coordinates of the centers of the circles. Let $x_1, x_2, y_1,$ and y_2 be the x and y coordinates of the centers of the circles. (It does not matter which circle is labeled 1 or 2; the end result will be the same.) Then the normal vector \vec{n} is:

$$\vec{n} = \langle x_2 - x_1, y_2 - y_1 \rangle$$

Next, find the unit vector of \vec{n} , which we will call \vec{un} .

This is done by dividing by the magnitude of \vec{n} :

$$\vec{un} = \frac{\vec{n}}{|\vec{n}|} = \frac{\vec{n}}{\sqrt{n_x^2 + n_y^2}}$$

Next we need the unit tangent vector. This is easy to find from the unit normal vector. Just make the x component of the unit tangent vector equal to the negative of the y component of the unit normal vector, and make the y component of the unit tangent vector equal to the x component of the unit normal vector: $\vec{ut} = \langle -un_y, un_x \rangle$

2. Create the initial (before the collision) velocity vectors, v_1 and v_2 . These are just the x and y components of the velocities put into vectors: $\vec{v}_1 = \langle v_{1x}, v_{1y} \rangle$ (and similarly for \vec{v}_2). Note that this step really isn't necessary if the velocities are already represented as vectors. This step is needed only if the velocities are initially represented as separate x and y values.
3. Keep in mind that after the collision the tangential component of the velocities is unchanged and the normal component of the velocities can be found using the one-dimensional collision formulas presented earlier. So we need to resolve the velocity vectors, \vec{v}_1 and \vec{v}_2 , into normal and tangential components. To do this, project the velocity vectors onto the unit normal and unit tangent vectors by computing the dot product. Let v_{1n} be the scalar (plain number, not a vector) velocity of object 1 in the normal direction. Let v_{1t} be the scalar velocity of object 1 in the tangential direction. Similarly, let v_{2n} and v_{2t} be for object 2. These values are found by

projecting the velocity vectors onto the unit normal and unit tangent vectors, which is done by taking the dot product:

$$v_{1n} = \vec{u}\vec{n} \cdot \vec{v}_1 \quad v_{1t} = \vec{u}\vec{t} \cdot \vec{v}_1 \quad v_{2n} = \vec{u}\vec{n} \cdot \vec{v}_2 \quad v_{2t} = \vec{u}\vec{t} \cdot \vec{v}_2$$

4. Find the new tangential velocities (after the collision). This is the simplest step of all. The tangential components of the velocity do not change after the collision because there is no force between the circles in the tangential direction during the collision. So, the new tangential velocities are simply equal to the old ones:

$$v'_{1t} = v_{1t} \quad v'_{2t} = v_{2t}$$

Remember that the apostrophe after the variable name means “after the collision.”

5. Find the new normal velocities. This is where we use the one-dimensional collision formulas. The velocities of the two circles along the normal direction are perpendicular to the surfaces of the circles at the point of collision, so this really is a one-dimensional collision.

$$v'_{1n} = \frac{v_{1n}(m_1 - m_2) + 2m_2 v_{2n}}{m_1 + m_2} \quad v'_{2n} = \frac{v_{2n}(m_2 - m_1) + 2m_1 v_{1n}}{m_1 + m_2}$$

6. Convert the scalar normal and tangential velocities into vectors. This is easy—just multiply the unit normal vector by the scalar normal velocity and you get a vector which has a direction that is normal to the surfaces at the point of collision and which has a magnitude equal to the normal component of the velocity. It is similar for the tangential component.

$$\vec{v}'_{1n} = v'_{1n} \cdot \vec{u}\vec{n} \quad \vec{v}'_{1t} = v'_{1t} \cdot \vec{u}\vec{t} \quad \vec{v}'_{2n} = v'_{2n} \cdot \vec{u}\vec{n} \quad \vec{v}'_{2t} = v'_{2t} \cdot \vec{u}\vec{t}$$

7. Find the final velocity vectors by adding the normal and tangential components for each object:

$$\vec{v}'_1 = \vec{v}'_{1n} + \vec{v}'_{1t} \quad \vec{v}'_2 = \vec{v}'_{2n} + \vec{v}'_{2t}$$

Now we have the final (after collision) velocity of each object as a vector.

To see these formulas in action, check out:

[Bouncescope](http://www.vobarian.com/bouncescope/) (<http://www.vobarian.com/bouncescope/>). This free program allows you to simulate lots of balls bouncing around elastically. The mass, size, initial velocity, and color of each ball can be customized, or you can add batches of randomly generated balls. The source code is provided.

To check for updated versions of this document, visit:

<http://www.vobarian.com/collisions/>

Vobarian Software homepage:

<http://www.vobarian.com/>

Feedback and comments are welcome via the e-mail address listed on the Vobarian Software website.

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